

BRIDGE SURFACES WITH THE TOPOLOGICAL MINIMALITY PRESERVED BY PERTURBATION

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ABSTRACT. We show that except for $n = 2$ if a bridge surface for a knot is an index n topologically minimal surface, then after a perturbation it is still topologically minimal with index at most $n + 1$.

1. INTRODUCTION

For a closed 3-manifold M , a *Heegaard splitting* $M = V^+ \cup_S V^-$ is a decomposition of M into two handlebodies V^+ and V^- with $\partial V^+ = \partial V^- = S$.

Let K be a knot in M . The notion of Heegaard splitting can be extended to the pair (M, K) . Suppose that $V^\pm \cap K$ is a collection of n boundary parallel arcs a_1^\pm, \dots, a_n^\pm in V^\pm . Each a_i^\pm is called a *bridge*. The decomposition $(M, K) = (V^+, V^+ \cap K) \cup_S (V^-, V^- \cap K)$ is called a *bridge splitting* of (M, K) , and we say that K is in n -bridge position with respect to S . By a *bridge surface*, we mean $S - K$.

Compressing disks for the bridge surface $S - K$ in $M - K$ and the information on how they intersect enable us to understand topological properties of (M, K) . The *disk complex* $\mathcal{D}(F)$ of a surface F embedded in a 3-manifold is a simplicial complex defined as follows.

- Vertices of $\mathcal{D}(F)$ are isotopy classes of compressing disks for F .
- A collection of $k + 1$ vertices forms a k -simplex if there are representatives for each vertex that are pairwise disjoint.

A surface is *incompressible* if there are no compressing disks, so the disk complex of an incompressible surface is empty. A surface F is *strongly irreducible* if F compresses to both sides and every compressing disk for F on one side intersects every compressing disk on the opposite side. So the disk complex of a strongly irreducible surface is disconnected. Extending these notions, Bachman defined topologically minimal surfaces [1], which can be regarded as topological analogues of (geometrically) minimal surfaces.

A surface F is *topologically minimal* if $\mathcal{D}(F)$ is empty or $\pi_i(\mathcal{D}(F))$ is non-trivial for some i . The *topological index* of F is 0 if $\mathcal{D}(F)$ is empty, and the smallest n such that $\pi_{n-1}(\mathcal{D}(F))$ is non-trivial, otherwise. Equivalently, an index n topologically minimal surface F has an $(n - 2)$ -connected $\mathcal{D}(F)$ and $\pi_{n-1}(\mathcal{D}(F))$ is non-trivial. Topologically minimal surfaces have nice properties, e.g. if an irreducible manifold contains an incompressible surface

and a topologically minimal surface, then the two surfaces can be isotoped so that any intersection loop is essential in both surfaces.

A *perturbation* is an operation on a bridge splitting that perturbs K near a point of $K \cap S$ so that a new local minimum and an adjacent local maximum is created. The two new bridges admit cancelling disks that intersect in one point. See Figure 1. In this paper, we show that if a bridge surface is topologically minimal, then a perturbation preserves topological minimality, except for one case. More precisely,

Theorem 1.1. *If a bridge surface for a knot is an index $n (\neq 2)$ topologically minimal surface, then after a perturbation it is still topologically minimal with index at most $n + 1$.*

The main idea of the proof is to construct a retraction from the disk complex of a bridge surface to a space whose homotopy group is non-trivial as in [2] and [4]. We conjecture that the topological index of the perturbed bridge surface in Theorem 1.1 is $n + 1$.

If we use (reduced) homology groups instead of homotopy groups in the definition of topological index, then Theorem 1.1 holds for all n .

Definition 1.2. *A surface F is strongly topologically minimal if $\mathcal{D}(F)$ is empty or $\tilde{H}_i(\mathcal{D}(F))$ is non-trivial for some i . The strong topological index of F is 0 if $\mathcal{D}(F)$ is empty, and the smallest n such that $\tilde{H}_{n-1}(\mathcal{D}(F))$ is non-trivial, otherwise.*

Theorem 1.3. *If a bridge surface for a knot is an index n strongly topologically minimal surface, then after a perturbation it is still strongly topologically minimal with index at most $n + 1$.*

2. PROOF OF THEOREM 1.1

Let M be decomposed into two handlebodies V^+ and V^- with common boundary S , and let K be a knot in bridge position with respect to S . Let \overline{K} be a knot obtained from K by a perturbation with cancelling disks D and E in V^+ and V^- respectively. See Figure 1. Let \overline{D} denote a disk in $V^+ - K$ such that $\partial \overline{D} = \partial N(D \cap S)$, where $N(D \cap S)$ is a neighborhood of $D \cap S$ taken in S , and similarly let \overline{E} denote a disk in $V^- - K$ such that $\partial \overline{E} = \partial N(E \cap S)$. See Figure 2. Obviously, we may assume that a disk disjoint from D (resp. E) is also disjoint from \overline{D} (resp. \overline{E}).

Lemma 2.1. *We can naturally embed $\mathcal{D}(S - K)$ into $\mathcal{D}(S - \overline{K})$.*

Proof. We can identify a neighborhood $N(K)$ of K with a neighborhood $N(\overline{K} \cup D \cup E)$ of $\overline{K} \cup D \cup E$ since $|D \cap E| = 1$. See Figure 3. Then a compressing disk C for $S - K$ in $V^\pm - N(K)$ corresponds to a compressing disk C' in $V^\pm - N(\overline{K} \cup D \cup E)$, and C' is also a compressing disk for $S - \overline{K}$ in $V^\pm - N(\overline{K})$. Hence, by this embedding we may regard $\mathcal{D}(S - K)$ as a subcomplex of $\mathcal{D}(S - \overline{K})$. \square

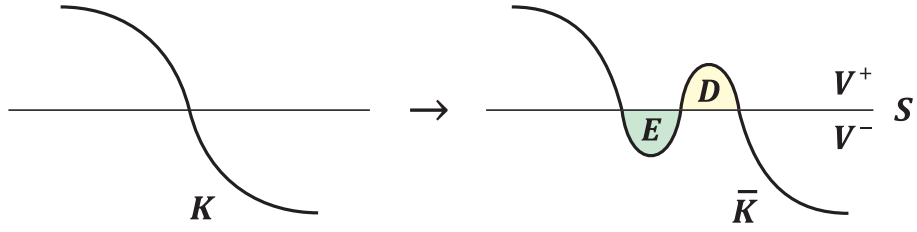
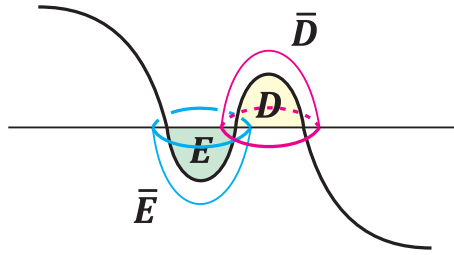
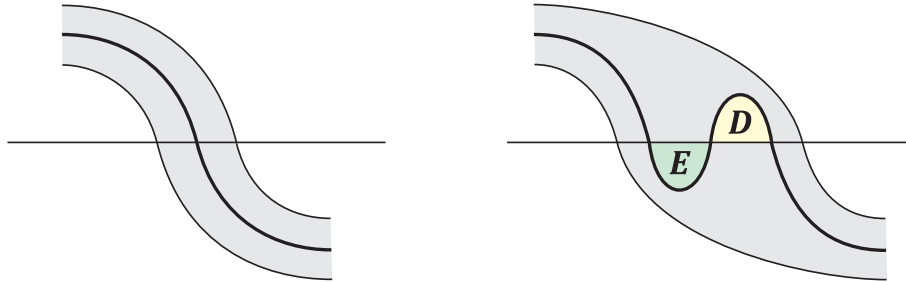


FIGURE 1. A perturbation

FIGURE 2. \bar{D} and \bar{E} FIGURE 3. $N(K)$ and $N(\bar{K} \cup D \cup E)$

We give a partition of the set of vertices of $\mathcal{D}(S - \bar{K})$ as follows.

- (1) $\mathcal{E}_1 = \{\bar{E}\}$
- (2) $\mathcal{E}_2 = \{\text{compressing disks in } V^- - \bar{K} \text{ other than } \bar{E}\}$
- (3) $\mathcal{E}_3 = \{\text{compressing disks in } V^+ - \bar{K} \text{ that are disjoint from } E\}$
- (4) $\mathcal{E}_4 = \{\text{compressing disks in } V^+ - \bar{K} \text{ that intersect } E\}$

The four collections $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$ are mutually disjoint and any compressing disk belongs to one of the collections. A compressing disk of $\mathcal{D}(S - K)$ (as a subcomplex of $\mathcal{D}(S - \bar{K})$ by Lemma 2.1) in $V^- - \bar{K}$ (resp. $V^+ - \bar{K}$) belongs to \mathcal{E}_2 (resp. \mathcal{E}_3). The disk \bar{D} belongs to \mathcal{E}_4 .

We define a map r_0 from the set of vertices of $\mathcal{D}(S - \bar{K})$ to the union of the set of vertices of $\mathcal{D}(S - K)$ and $\{\bar{D}, \bar{E}\}$. Since \bar{D} and \bar{E} are disjoint from the compressing disks of $\mathcal{D}(S - K)$, we can consider a simplicial complex

$\text{Sus}(\mathcal{D}(S - K))$, which is the suspension of $\mathcal{D}(S - K)$ over $\{\overline{D}, \overline{E}\}$. It will be shown later that the map r_0 extends to a retraction of $\mathcal{D}(S - \overline{K})$ onto $\text{Sus}(\mathcal{D}(S - K))$.

(1) We define $r_0(\overline{E})$ to be \overline{E} .

(2) Let $C \subset V^- - \overline{K}$ be a compressing disk other than \overline{E} . Suppose that C has nonempty minimal intersection (in its isotopy class) with E . We may assume that $C \cap E$ consists of arc components by standard innermost disk argument. Choose any outermost arc of $C \cap E$ in C and let Δ be the corresponding outermost disk. A disk surgery of E along Δ yields a bridge disk and a compressing disk C_1 disjoint from E . In case that C does not intersect E , let $C_1 = C$. So in any case, $C_1 \cap E = \emptyset$.

If there is any intersection point of $\partial C_1 \cap D$, let q be the point of $\partial C_1 \cap D$ which is closest to $p = D \cap E$ in the arc $D \cap S$. Then we connect a copy of \overline{E} to C_1 by a band along \overline{pq} as in Figure 4 and get a new disk C_2 with $|\partial C_2 \cap D| < |\partial C_1 \cap D|$. We perform this operation for all intersection points of $\partial C_1 \cap D$, and let C' be the resulting disk with $\partial C' \cap D = \emptyset$. In fact, C' is isotopic to C_1 if we remove E by isotopy as in Figure 5. We define $r_0(C)$ to be C' . Note that $C' \cap (D \cup E) = \emptyset$, hence C' can be regarded as a disk in $\mathcal{D}(S - K)$.

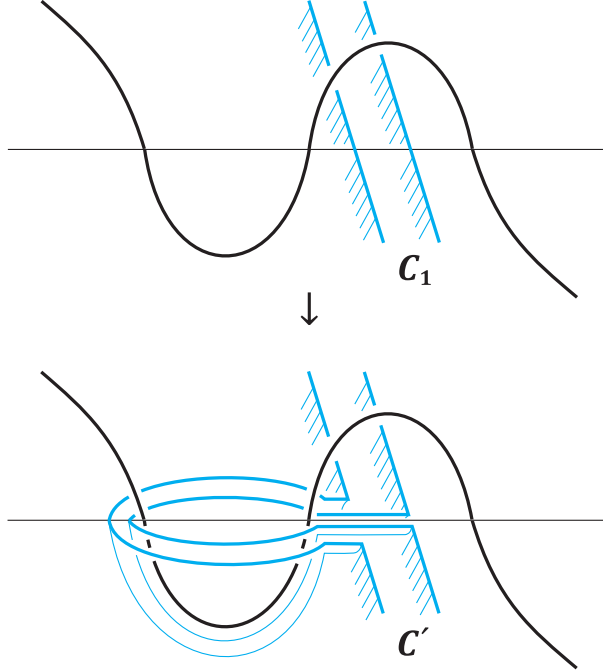


FIGURE 4.

(3) Let $C \subset V^+ - \overline{K}$ be a compressing disk that is disjoint from E . Suppose that C intersects D . We may assume that $C \cap D$ consists of arc

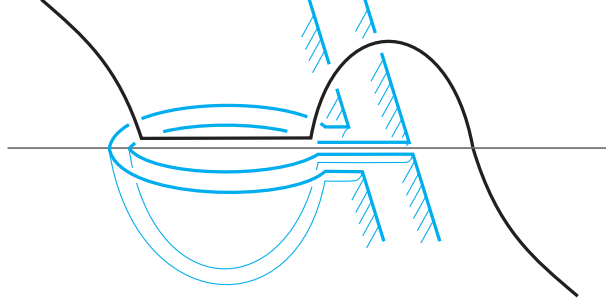


FIGURE 5.

components. For every arc α of $C \cap D$, we cut off C by α and reglue the two resulting subdisks along a slightly detouring band passing through the bridge disk ($\neq D$) adjacent to E , as in Figure 6. Even though arcs of $C \cap D$ are nested in D , this operation is possible. Let C' be the resulting disk obtained from C . If C does not intersect D , let $C' = C$. The disk C' is isotopic to C if we remove E by isotopy. We define $r_0(C)$ to be C' . Note that $C' \cap (D \cup E) = \emptyset$, hence C' can be regarded as a disk in $\mathcal{D}(S - K)$.

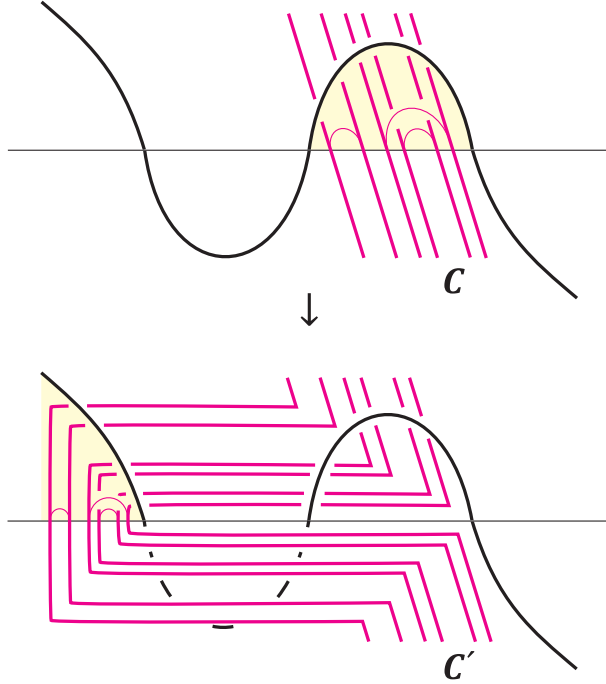


FIGURE 6.

(4) Let $C \subset V^+ - \overline{K}$ be a compressing disk that intersects E . We define $r_0(C)$ to be \overline{D} .

The restriction of r_0 to the set of vertices of $\text{Sus}(\mathcal{D}(S - K))$ is the identity map. Next we show that r_0 can be extended to a continuous map r_1 from the 1-skeleton of $\mathcal{D}(S - \overline{K})$ to the 1-skeleton of $\text{Sus}(\mathcal{D}(S - K))$. Then r_1 will be a retraction.

Lemma 2.2. *The map r_0 extends to a continuous map r_1 from the 1-skeleton of $\mathcal{D}(S - \overline{K})$ to the 1-skeleton of $\text{Sus}(\mathcal{D}(S - K))$.*

Proof. It suffices to show that for disjoint compressing disks C_1 and C_2 of $\mathcal{D}(S - \overline{K})$, either $r_0(C_1)$ and $r_0(C_2)$ are disjoint, or $r_0(C_1) = r_0(C_2)$. Without loss of generality, there are the following cases to consider.

Case 1. $C_1 \in \mathcal{E}_1$ and $C_2 \in \mathcal{E}_2$.

Since $r_0(C_1) = r_0(\overline{E}) = \overline{E}$ and $r_0(C_2)$ is a disk in $\mathcal{D}(S - K)$, $r_0(C_1)$ and $r_0(C_2)$ are disjoint.

Case 2. $C_1 \in \mathcal{E}_1$ and $C_2 \in \mathcal{E}_3$.

The disk $r_0(C_2)$ is a disk in $\mathcal{D}(S - K)$. So similarly as Case 1, $r_0(C_1)$ and $r_0(C_2)$ are disjoint.

Case 3. $C_1 \in \mathcal{E}_2$ and $C_2 \in \mathcal{E}_3$.

Suppose that C_1 intersects E . The operations in the definition of $r_0(C_1)$ are done in two steps. In the step of disk surgery of E along outermost disk, the resulting disk is disjoint from C_2 because C_2 is disjoint from E .

The remaining banding operations for $r_0(C_1)$ and $r_0(C_2)$ result disjoint $r_0(C_1)$ and $r_0(C_2)$. See Figure 7 for an example.

Case 4. $C_1 \in \mathcal{E}_2$ and $C_2 \in \mathcal{E}_4$.

Since $r_0(C_1)$ is a disk in $\mathcal{D}(S - K)$ and $r_0(C_2) = \overline{D}$, $r_0(C_1)$ and $r_0(C_2)$ are disjoint.

Case 5. $C_1 \in \mathcal{E}_3$ and $C_2 \in \mathcal{E}_4$.

Since $r_0(C_1)$ is a disk in $\mathcal{D}(S - K)$ and $r_0(C_2) = \overline{D}$, $r_0(C_1)$ and $r_0(C_2)$ are disjoint.

Case 6. Both $C_1, C_2 \in \mathcal{E}_2$.

We can see that both the disk surgery and banding operations of r_0 result disjoint $r_0(C_1)$ and $r_0(C_2)$, or $r_0(C_1) = r_0(C_2)$.

Case 7. Both $C_1, C_2 \in \mathcal{E}_3$.

We can see that $r_0(C_1)$ and $r_0(C_2)$ are disjoint, or $r_0(C_1) = r_0(C_2)$.

Case 8. Both $C_1, C_2 \in \mathcal{E}_4$.

In this case, $r_0(C_1) = r_0(C_2) = \overline{D}$. □

Since higher dimensional simplices of $\mathcal{D}(S - \overline{K})$ are determined by 1-simplices, r_1 extends to a retraction $r : \mathcal{D}(S - \overline{K}) \rightarrow \text{Sus}(\mathcal{D}(S - K))$.

Suppose that $S - K$ is an index n topologically minimal surface. First, consider the case of $n = 0$. Then the incompressibility of $S - K$ implies that the genus of S is 0 and K is in 1-bridge position, i.e. K is a 1-bridge unknot in S^3 . A perturbation of K yields a 2-bridge splitting for the unknot, which is strongly irreducible, hence index 1. So Theorem 1.1 holds when $n = 0$. Now we assume that $n = 1$ or $n \geq 3$.

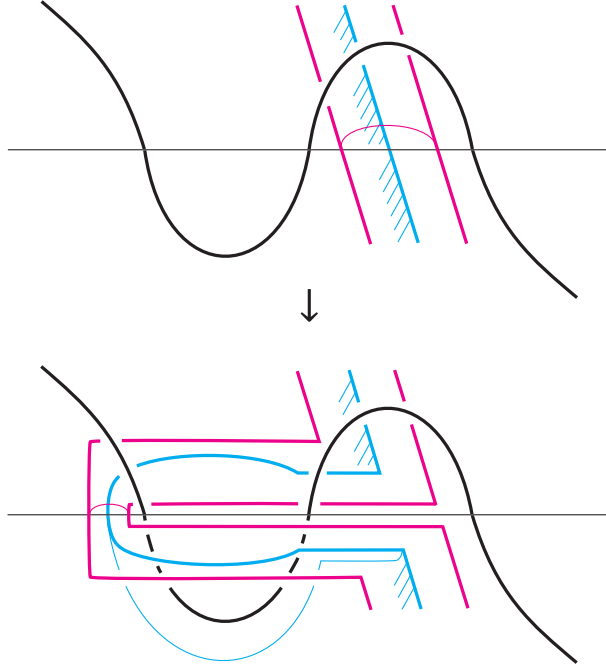


FIGURE 7.

Claim 1. $\pi_n(\text{Sus}(\mathcal{D}(S - K))) \neq 1$.

Proof of Claim 1. Suppose that $n = 1$. Since $S - K$ is an index 1 topologically minimal surface, $\pi_0(\mathcal{D}(S - K)) \neq 1$, i.e. $\mathcal{D}(S - K)$ is disconnected. In fact, it has two contractible components—the subcomplexes spanned by compressing disks in $V^+ - K$ and $V^- - K$. Then the fundamental group of the suspension of $\mathcal{D}(S - K)$ is infinite cyclic and the claim holds.

So we may assume that $n \geq 3$. Since $S - K$ is an index n topologically minimal surface, $\mathcal{D}(S - K)$ is $(n - 2)$ -connected and $\pi_{n-1}(\mathcal{D}(S - K)) \neq 1$. It is known that the suspension map

$$\pi_i(\mathcal{D}(S - K)) \rightarrow \pi_{i+1}(\text{Sus}(\mathcal{D}(S - K)))$$

is an isomorphism for $i < 2(n - 1) - 1$ and a surjection for $i = 2(n - 1) - 1$. (See e.g. [3, Corollary 4.24].) Hence $\pi_n(\text{Sus}(\mathcal{D}(S - K))) \neq 1$. \square

The retraction r induces a surjective map $r_* : \pi_n(\mathcal{D}(S - \overline{K})) \rightarrow \pi_n(\text{Sus}(\mathcal{D}(S - K)))$. So $\pi_n(\mathcal{D}(S - \overline{K})) \neq 1$, and the topological index of $S - \overline{K}$ is at most $n + 1$.

3. WHEN $n = 2$ AND THE NON-TRIVIAL HOMOLOGY CONDITION

In this section we investigate the case of $n = 2$ in detail. Suppose that $S - K$ is an index 2 topologically minimal surface. Then $\pi_0(\mathcal{D}(S - K)) = 1$ and

$\pi_1(\mathcal{D}(S-K)) \neq 1$. The suspension map $\pi_1(\mathcal{D}(S-K)) \rightarrow \pi_2(\text{Sus}(\mathcal{D}(S-K)))$ is a surjection and does not guarantee that $\pi_2(\text{Sus}(\mathcal{D}(S-K)))$ is non-trivial.

The suspension $\text{Sus}(\mathcal{D}(S-K))$ is a union of an upper cone A and a lower cone B and $A \cap B \simeq \mathcal{D}(S-K)$. By the van Kampen theorem, $\pi_1(\text{Sus}(\mathcal{D}(S-K))) = 1$ because $\pi_1(A) = \pi_1(B) = 1$. Applying the Mayer-Vietoris sequence, the long exact sequence

$$\cdots \rightarrow \tilde{H}_{i+1}(A) \oplus \tilde{H}_{i+1}(B) \rightarrow \tilde{H}_{i+1}(\text{Sus}(\mathcal{D}(S-K))) \rightarrow \tilde{H}_i(\mathcal{D}(S-K)) \rightarrow \tilde{H}_i(A) \oplus \tilde{H}_i(B) \rightarrow \cdots$$

is exact. Since $\tilde{H}_i(A) = \tilde{H}_i(B) = 1$ for all i , we have

$$\tilde{H}_{i+1}(\text{Sus}(\mathcal{D}(S-K))) \simeq \tilde{H}_i(\mathcal{D}(S-K)).$$

In particular, $\tilde{H}_2(\text{Sus}(\mathcal{D}(S-K))) \simeq \tilde{H}_1(\mathcal{D}(S-K))$. Since $\pi_0(\text{Sus}(\mathcal{D}(S-K))) = 1$ and $\pi_1(\text{Sus}(\mathcal{D}(S-K))) = 1$, by the Hurewicz theorem $\pi_2(\text{Sus}(\mathcal{D}(S-K))) \simeq \tilde{H}_2(\text{Sus}(\mathcal{D}(S-K)))$. So we conclude that $\pi_2(\text{Sus}(\mathcal{D}(S-K))) \simeq \tilde{H}_1(\mathcal{D}(S-K))$. The group $\tilde{H}_1(\mathcal{D}(S-K))$ is isomorphic to $\pi_1(\mathcal{D}(S-K))/C$, where C is the commutator subgroup of $\pi_1(\mathcal{D}(S-K))$. Hence if $\pi_1(\mathcal{D}(S-K))$ is not equal to C , i.e. if $\pi_1(\mathcal{D}(S-K))$ is not a perfect group, then $\pi_2(\text{Sus}(\mathcal{D}(S-K))) \neq 1$ and the topological index of $S - \overline{K}$ is at most 3.

Proof of Theorem 1.3. Suppose that $S - K$ is an index n strongly topologically minimal surface. The case of $n = 0$ is similar to the proof of Theorem 1.1. So we assume that $n \geq 1$. By definition, $\tilde{H}_{n-1}(\mathcal{D}(S-K)) \neq 1$. The above mentioned isomorphism $\tilde{H}_n(\text{Sus}(\mathcal{D}(S-K))) \rightarrow \tilde{H}_{n-1}(\mathcal{D}(S-K))$ implies that $\tilde{H}_n(\text{Sus}(\mathcal{D}(S-K))) \neq 1$. The retraction r induces a surjective map $r_* : \tilde{H}_n(\mathcal{D}(S - \overline{K})) \rightarrow \tilde{H}_n(\text{Sus}(\mathcal{D}(S-K)))$. So $\tilde{H}_n(\mathcal{D}(S - \overline{K})) \neq 1$, and the strong topological index of $S - \overline{K}$ is at most $n + 1$. \square

Acknowledgements. The author would like to thank Jae Choon Cha and Daewoong Lee for helpful comments.

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